

Semilinear Periodic-Parabolic Equations with Nonlinear Boundary Conditions

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Recently much work has been devoted to periodic-parabolic equations with linear homogeneous boundary conditions. However, very little has been accomplished in the literature for periodic-parabolic problems with nonlinear boundary

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lower solutions are given. Fractional order function spaces, Ehrling–Gagliardo–Nirenberg and Lions–Peetre–Calderón type interpolation inequalities for functions in (anisotropic) Sobolev–Slobodeckii spaces play an important role in the obtainment of *a priori* boundary and interior estimates. In proving our existence results we make use of topological degree techniques and regularity results for linear parabolic partial differential equations under linear nonhomogeneous boundary conditions. We also indicate how one can obtain minimal and maximal time-periodic solutions to parabolic problems with nonlinear boundary conditions.

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1. INTRODUCTION

In this paper we consider second order parabolic partial differential equations

$$\begin{aligned} \frac{\partial u}{\partial t} + A(x, t, D)u &= f(x, t, u, \nabla u) & \text{in } \Omega \times [0, T], \\ \frac{\partial u}{\partial \eta} &= h(x, t, u) & \text{on } \partial\Omega \times [0, T], \\ u(x, 0) &= u(x, T) & \text{on } \bar{\Omega}, \end{aligned} \quad (1)$$

where Ω is a bounded domain, $A(x, t, D)$ is, for each t , a second order uniformly (strongly) elliptic partial differential operator with time-periodic

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coefficients, η is an outward pointing time-periodic vector field, and f and h are (possibly) nonlinear functions which are time-periodic such that h is locally Lipschitz continuous and f is locally Hölder continuous with quadratic growth in ∇u .

Many papers are concerned with the steady-state or elliptic version of problem (1) with smooth function f (see e.g. [2, 4, 7, 8, 11, 15, 17, 24, 25, 28, 29] and references therein). Of course the steady-state problem is a special case of Eq. (1) (see e.g. [4, 19, 21]). Recently, several papers have been devoted to initial-boundary value problems for parabolic equations with nonlinear boundary conditions and smooth functions f and h (see e.g. [5, 7, 28, 32] and references therein).

Likewise, periodic-parabolic problems with homogeneous and autonomous linear boundary conditions have been studied by many authors (see e.g. [3, 9, 10, 14, 16, 18, 19, 21, 31] and references therein). For a discussion on how periodic-parabolic problems occur in applications, the reader is referred to the paper [9] and the book [16], among others.

However, very little has been accomplished in the literature for periodic-parabolic problems with nonlinear boundary conditions. Problems with nonlinear boundary conditions naturally occur in applications (see e.g. [12]), and time-periodic solutions are of interest since they arise in the dynamics or asymptotic behavior of solutions to initial-boundary value problems for parabolic partial differential equations (see e.g. [3, 4, 12, 16, 19, 21, 29]); recall that equilibrium solutions are also time-periodic solutions (see e.g. [29]).

Since the function f is not (necessarily) required to be locally Lipschitz continuous (or locally monotone) in its third variable, uniformly in the other variables in bounded sets, uniqueness of solutions to initial-boundary value problems for parabolic equations is not assured. This usually precludes the use of discrete-time semi-groups or Poincaré operator methods for the obtainment of time-periodic solutions. Also, owing to the non-autonomous character of the boundary conditions, the domains of the generators of the semi-groups would be time-dependent, albeit periodically. (We refer to the paper [5] for a discussion of the difficulties encountered in studying initial-boundary value problems for parabolic evolution equations and nonlinear boundary conditions in the framework of “time-dependent semigroups” methods or “variation-of-parameters” formula associated with the evolution operator.)

It is the main purpose of this paper to study existence questions for periodic-parabolic problems with nonlinear boundary conditions provided ordered upper and lower solutions are given. Our approach mainly relies on *a priori* estimates for periodic-parabolic problems with nonlinear boundary conditions and Leray–Schauder degree arguments. Therefore, our approach to existence questions for Eq. (1) is different from those

previously used (such as in [3, 16, 19, 31]) for periodic-parabolic problems with homogeneous and autonomous linear boundary conditions. Our *a priori* estimates and existence results include those contained in the papers [3, 9, 10, 14, 16, 18, 19, 21, 31].

As aforementioned, the obtainment of *a priori* estimates for parabolic problems with quadratic growth in ∇u (for the nonlinear function f) and nonlinear boundary conditions is of particular importance. These estimates are proved in Section 3 for initial-boundary value and time-periodic problems with nonhomogeneous and nonautonomous linear boundary conditions as well as for problems with nonlinear boundary conditions. Along the way, we derive interpolation inequalities of Ehrling–Gagliardo–Nirenberg and Lions–Peetre–Calderón type with respect to the norm of functions in (anisotropic) Sobolev–Slobodeckii spaces of the form $W_p^{1, 1/2}(\Omega \times [0, T])$ (Lemma 3.2). These interpolation inequalities are especially needed for problems with nonlinear boundary conditions.

Obviously the above boundary conditions include Neumann and regular oblique derivative linear boundary conditions. Let us mention that the approach developed herein does also apply to the periodic-parabolic problem with nonhomogeneous and nonautonomous Dirichlet boundary conditions

$$\frac{\partial u}{\partial t} + A(x, t, D) u = f(x, t, u, \nabla u) \quad \text{in } \Omega \times [0, T],$$

$$u = \varphi(x, t) \quad \text{on } \partial\Omega \times [0, T],$$

$$u(x, 0) = u(x, T) \quad \text{on } \bar{\Omega},$$

where $\phi \in W_p^{2-1/p, 1-1/2p}(\partial\Omega \times [0, T])$ (an appropriate trace-space) with $\varphi(x, 0) = \varphi(x, T)$ for all $x \in \partial\Omega$. However, we have elected to present our results only for problems with nonlinear boundary conditions since this is the case we are mainly concerned with. (Actually, for time-periodic smooth $\varphi \in C^{2+\mu, 1+\mu/2}(\partial\Omega \times [0, T])$, the periodic-parabolic problem with nonhomogeneous Dirichlet boundary conditions can be studied by using a device used in [4, pp. 291–292] or [29] to reduce the problem to one with homogeneous Dirichlet boundary conditions and our approach developed herein.)

The paper is organized as follows. In Section 2 we collect the explicit conditions imposed on the data; that is, on the domain Ω , the operator $A(x, t, D)$, the vector field η , and the nonlinearities f and h . Furthermore, we give the definitions of what we mean by a classical solution, and upper and lower solutions for the periodic-parabolic problem (1). Section 3 is

devoted to obtaining *a priori* estimates for initial-boundary value problems and periodic-parabolic problems with nonlinear boundary conditions (Propositions 3.1–3.3). In Section 4, we combine results obtained in Section 3 and Leray–Schauder degree arguments to prove existence results for (classical) solutions to periodic-parabolic problem (1) provided ordered upper and lower solutions are given. In that regard, we transform Eq. (1) into a periodic-parabolic problem with regular oblique derivative nonlinear boundary conditions to which we apply topological degree techniques (Theorem 4.1). Furthermore, assuming only mild regularity conditions on the nonlinear function h , we prove that the classical solution obtained in Theorem 4.1 actually is a regular solution (Theorem 4.2). To conclude the paper, we discuss a more general concept of upper and lower solutions (Remark 4.1) and indicate how one can derive minimal and maximal solutions to Eq. (1) from our existence result.

2. PRELIMINARIES

Let $\Omega \subset \mathbb{R}^N$ be a bounded domain whose boundary $\partial\Omega$ is an $(N-1)$ -dimensional submanifold of class $C^{2+\mu}$, $0 < \mu < 1$, such that Ω lies locally on one side of $\partial\Omega$, and let $I = [0, T]$ with $T > 0$.

Let $A(x, t, D)$ and L be the second order partial differential operators given by

$$A(x, t, D)u = - \sum_{i,j=1}^N a_{ij}(x, t) \frac{\partial^2 u}{\partial x_i \partial x_j} + \sum_{i=1}^N a_i(x, t) \frac{\partial u}{\partial x_i} + a_0(x, t)u,$$

$$Lu = \frac{\partial u}{\partial t} + A(x, t, D)u$$

where $a_{ij} = a_{ji}$, and L is uniformly (strongly) parabolic; that is, there exists a constant $\delta > 0$ such that

$$\sum_{i,j=1}^N a_{ij}(x, t) \xi_i \xi_j \geq \delta |\xi|^2 \quad \text{for all } (x, t) \in \bar{\Omega} \times I \text{ and all } \xi \in \mathbb{R}^N. \quad (2)$$

We assume that the coefficients $a_{ij}: \bar{\Omega} \times I \rightarrow \mathbb{R}$, $a_i: \bar{\Omega} \times I \rightarrow \mathbb{R}$, $1 \leq i, j \leq N$, and $a_0: \bar{\Omega} \times I \rightarrow \mathbb{R}$ belong to the Banach space of μ -Hölder continuous and T -periodic functions $C_T^{\mu, \mu/2}(\bar{\Omega} \times I) = \{u \in C^{\mu, \mu/2}(\bar{\Omega} \times I) : u(x, 0) = u(x, T) \text{ for all } x \in \bar{\Omega}\}$ with

$$a_0(x, t) \geq 0 \quad \text{for all } (x, t) \in \bar{\Omega} \times I. \quad (3)$$

We use the metric $d((x, t), (y, s)) = (|x - y|^2 + |s - t|)^{1/2}$ for the computation of the Hölder (and Lipschitz) constants. Throughout this paper $\bar{\Omega}$ denotes the closure of Ω in \mathbb{R}^N .

Let $f: \bar{\Omega} \times I \times \mathbb{R} \times \mathbb{R}^N \rightarrow \mathbb{R}$ be a locally μ -Hölder continuous function which is T -periodic in t ; that is, $f(x, 0, u, \xi) = f(x, T, u, \xi)$ for all $(x, u, \xi) \in \bar{\Omega} \times \mathbb{R} \times \mathbb{R}^N$, and for each $(u, \xi) \in \mathbb{R} \times \mathbb{R}^N$ there exist a neighborhood $V \subset \mathbb{R} \times \mathbb{R}^N$ of (u, ξ) and a number $K > 0$ such that

$$|f(x, t, v, p) - f(y, s, w, q)| \leq K(|x - y|^2 + |t - s| + |v - w|^2 + |p - q|^2)^{\mu/2} \quad (4)$$

for all $(x, t, v, p), (y, s, w, q) \in \bar{\Omega} \times I \times V$.

We assume that there exists a continuous function $c: \mathbb{R}_+ \rightarrow \mathbb{R}_+$ where $\mathbb{R}_+ = [0, \infty)$ such that

$$|f(x, t, u, \xi)| \leq c(\rho)(1 + |\xi|^2) \quad (5)$$

for every $\rho \geq 0$ and $(x, t, u, \xi) \in \bar{\Omega} \times I \times [-\rho, \rho] \times \mathbb{R}^N$. (If $N = 1$ we only assume that f is a continuous function which is T -periodic in t such that the (at most) quadratic growth condition (5) is satisfied.)

Let $h: \partial\Omega \times I \times \mathbb{R} \rightarrow \mathbb{R}$ be a locally Lipschitz continuous function which is T -periodic in t ; that is, $h(x, 0, u) = h(x, T, u)$ for all $(x, u) \in \partial\Omega \times \mathbb{R}$, and for each $u \in \mathbb{R}$ there exist a (closed) interval $U \subset \mathbb{R}$ about u and a number $M > 0$ such that

$$|h(x, t, v) - h(y, s, w)| \leq M(|x - y|^2 + |t - s| + |v - w|^2)^{1/2} \quad (6)$$

for all $(x, t, v), (y, s, w) \in \partial\Omega \times I \times U$. (Note that, strictly speaking, the function $h(x, \cdot, u)$ is $(1/2)$ -Hölder continuous in the variable t , uniformly for $(x, u) \in \partial\Omega \times U$.)

We are interested in the periodic-parabolic boundary value problem with nonlinear boundary conditions

$$\begin{aligned} Lu &= f(x, t, u, \nabla u) & \text{in } \Omega \times [0, T], \\ \frac{\partial u}{\partial \eta} &= h(x, t, u) & \text{for all } (x, t) \in \partial\Omega \times [0, T], \\ u(x, 0) &= u(x, T) & \text{for all } x \in \Omega, \end{aligned} \quad (7)$$

where $\eta \in C_T^{1+\mu, (1+\mu)/2}(\partial\Omega \times I, \mathbb{R}^N)$ is an outward pointing nowhere tangent (to $\partial\Omega$) vector field on $\partial\Omega \times I$ (see e.g. Ladyženskaja *et al.* [20, p. 318] for an explicit definition), $\partial u / \partial \eta$ denotes the directional derivative of u with respect to η , and ∇u is the gradient of u with respect to the space variable $x \in \mathbb{R}^N$ only.

We shall mainly be concerned with existence and regularity results for *classical* solutions to Eq. (7). A classical solution to Eq. (7) is a function $u \in C^{2,1}(\Omega \times [0, T]) \cap C^{1,0}(\bar{\Omega} \times I)$ which satisfies Eq. (7) pointwise.

A function α is called a *lower solution* for Eq. (7) if there exists a number $T_1 = T_1(\alpha) > T$ such that $\alpha \in C^{2,1}(\bar{\Omega} \times [0, T_1])$ and

$$\begin{aligned} L\alpha &\leq f(\cdot, \cdot, \alpha, \nabla \alpha) && \text{in } \Omega \times [0, T_1], \\ \frac{\partial \alpha}{\partial \eta} &\leq h(\cdot, \cdot, \alpha) && \text{on } \partial\Omega \times [0, T], \\ \alpha(\cdot, 0) &\leq \alpha(\cdot, T) && \text{on } \bar{\Omega}, \end{aligned} \quad (8)$$

A function $\beta \in C^{2,1}(\bar{\Omega} \times I)$ is called an *upper solution* for Eq. (7) if the above inequalities are reversed.

In Section 4 we shall prove that, given ordered lower and upper solutions α and β respectively, there is a (classical) solution to Eq. (7) lying between α and β . To do so, we shall need some *a priori* estimates.

3. A PRIORI ESTIMATES FOR PARABOLIC EQUATIONS

Throughout this section we shall assume that $p = (N+2)/(1-\mu)$, which implies that $p > N+2$ and that the function space $W_p^{2,1}(\Omega \times I)$ is continuously imbedded into $C^{1+\mu, (1+\mu)/2}(\bar{\Omega} \times I)$ (see e.g. Ladyženskaja *et al.* [20, p. 80, Lemma 3.3]). In what follows, we shall use the function spaces $W_p^{1,1/2}(\Omega \times I)$ and $W_p^{1-1/p, (1-1/p)/2}(\partial\Omega \times I)$ where $W_p^{1-1/p, (1-1/p)/2}(\partial\Omega \times I)$ denotes the space of traces of $W_p^{1,1/2}(\Omega \times I)$ -functions. We refer to Ladyženskaja *et al.* [20, Chap. I §1 and Chap. II §2–3] for explicit definitions and properties of function spaces used in this section. Note that, by the second part of Lemma 3.3 in [20, p. 80], $W_p^{1,1/2}(\Omega \times I)$ is continuously imbedded into $C^0(\bar{\Omega} \times I)$. The norm in the trace space $W_p^{1-1/p, (1-1/p)/2}(\partial\Omega \times I)$ is computed by using local coordinates for $\partial\Omega$ (see e.g. [20, pp. 81–82]).

Since we are interested in obtaining *a priori* estimates only, we shall assume in this section that each initial boundary value problem (respectively periodic boundary value problem) considered has at least one *classical* solution; by this we mean a function $u \in C^{2,1}(\Omega \times (0, T]) \cap C^{1,0}(\bar{\Omega} \times I)$ satisfying the given equations pointwise.

The *a priori* estimates obtained herein were motivated by similar results proved by Amann [4, Section 2] (whose results were inspired by those earlier on proved by Tomi in 1969 and v. Wahl in 1972 and 1973) for problems with homogeneous and autonomous linear boundary conditions. We stress, however, the fact that herein we prove the *a priori* estimates

for problems with (possibly) non-autonomous and *nonlinear* boundary conditions.

Let us mention that for results involving *initial* boundary value problems only, we may suppose, without loss of generality, that the vector field η and the coefficients in the operator L satisfy the above assumptions with the exception of the periodicity condition. The latter will be needed only when we will be dealing with periodic-parabolic boundary value problems.

LEMMA 3.1. *For every $a \in C^{v, v/2}(\bar{\Omega} \times I)$ where $0 < v \leq \mu$ is a given number, every $b \in W_p^{1-1/p, (1-1/p)/2}(\partial\Omega \times I)$ and every $d \in W_p^{2-2/p}(\Omega)$ satisfying the compatibility condition $d(x) + (\partial d / \partial \eta)(x) = b(x, 0)$ for all $x \in \partial\Omega$, one has that the unique classical solution $u \in C^{2,1}(\Omega \times (0, T]) \cap C^{1,0}(\Omega \times I)$ to the initial boundary value problem*

$$\begin{aligned} Lu + u &= a(x, t)(1 + |\nabla u|^2) && \text{for all } (x, t) \in \Omega \times (0, T], \\ u + \frac{\partial u}{\partial \eta} &= b(x, t) && \text{for all } (x, t) \in \partial\Omega \times (0, T], \\ u(x, 0) &= d(x) && \text{for all } x \in \bar{\Omega} \end{aligned} \quad (9)$$

is such that $u \in W_p^{2,1}(\Omega \times I)$. Moreover there is a function $\gamma_0: \mathbb{R}_+^3 \rightarrow \mathbb{R}_+$ depending only on \bar{L} , η , $\Omega \times I$, N and p such that

$$\begin{aligned} |u|_{W_p^{2,1}(\Omega \times I)} &\leq \gamma_0(|a|_{C^0(\bar{\Omega} \times I)}, |b|_{C^0(\partial\Omega \times I)}, |d|_{W_p^{2-2/p}(\Omega)}) \\ &\quad \times (1 + |b|_{W_p^{1-1/p, (1-1/p)/2}(\partial\Omega \times I)}), \end{aligned} \quad (10)$$

where γ_0 is increasing in each argument.

Proof. Since the right hand side of the differential equation in (9) is continuous on $\bar{\Omega} \times I$; that is, the function $a(\cdot, \cdot)(1 + |\nabla u|^2) \in C^0(\bar{\Omega} \times I) \subset L^p(\Omega \times I)$, it follows from [20, Chap. IV §9] that $u \in W_p^{2,1}(\Omega \times I)$.

To prove the uniqueness part of the statement, suppose u and $v \in W_p^{2,1}(\Omega \times I)$ are classical solutions to Eq. (9). Setting $w = u - v$, one has that $w \in W_p^{2,1}(\Omega \times I)$ is a solution to the initial boundary value problem

$$\begin{aligned} Lw - [a(x, t) \nabla(u + v)(x, t)] \cdot \nabla w + w &= 0 && \text{for all } (x, t) \in \Omega \times (0, T], \\ w + \frac{\partial w}{\partial \eta} &= 0 && \text{for all } (x, t) \in \partial\Omega \times (0, T], \\ w(x, 0) &= 0 && \text{for all } x \in \bar{\Omega}. \end{aligned} \quad (11)$$

Therefore, by uniqueness results for linear parabolic partial differential equations (see e.g. Ladyženskaja *et al.* [20, Chap. IV §9], it follows that $w = 0$ on $\bar{\Omega} \times I$.

Now, to prove the *a priori* estimate (10), we consider (for $\sigma \in [0, 1]$) the family of initial boundary value problems

$$\begin{aligned} Lu + u &= a(x, t)(\sigma + |\nabla u|^2) && \text{for all } (x, t) \in \Omega \times (0, T], \\ u + \frac{\partial u}{\partial \eta} &= \sigma b(x, t) && \text{for all } (x, t) \in \partial\Omega \times (0, T], \\ u(x, 0) &= \sigma d(x) && \text{for all } x \in \bar{\Omega} \end{aligned} \quad (12)$$

for which we assume, as aforementioned, the existence of a (necessarily unique) classical solution, denoted u_σ , which also is in $W_p^{2,1}(\Omega \times I)$ for each $\sigma \in [0, 1]$.

For $\sigma, \tau \in [0, 1]$, let us set

$$v = u_\sigma - u_\tau \quad \text{and} \quad R = |\sigma - \tau|(|a|_{C^0(\Omega \times I)} + |b|_{C^0(\partial\Omega \times I)} + |d|_{C^0(\bar{\Omega})}).$$

Then v is a classical solution to the initial boundary value problem

$$\begin{aligned} Lv - [a\nabla(u_\sigma + u_\tau)] \cdot \nabla v + v &= a(\sigma - \tau) && \text{for all } (x, t) \in \Omega \times (0, T], \\ v + \frac{\partial v}{\partial \eta} &= (\sigma - \tau) b && \text{for all } (x, t) \in \partial\Omega \times (0, T], \\ v(x, 0) &= (\sigma - \tau) d && \text{for all } x \in \bar{\Omega}. \end{aligned} \quad (13)$$

Since

$$\begin{aligned} L(R \pm v) - [a\nabla(u_\sigma + u_\tau)] \cdot \nabla(R \pm v) + (R \pm v) &= R \pm a(\sigma - \tau) \geq 0, \\ (R \pm v) + \frac{\partial}{\partial \eta}(R \pm v) &= R \pm (\sigma - \tau) b \geq 0, \\ (R \pm v)(x, 0) &= R \pm (\sigma - \tau) d \geq 0, \end{aligned}$$

it follows from the maximum principle for linear parabolic partial differential equations (see e.g. Ladyženskaja *et al.* [20, pp. 15–16, Lemma 2.2]) that $R \pm v \geq 0$, and hence

$$|v|_{C^0(\bar{\Omega} \times I)} \leq R. \quad (14)$$

This immediately implies that

$$|u_\tau|_{C^0(\bar{\Omega} \times I)} \leq \tau(|a|_{C^0(\bar{\Omega} \times I)} + |b|_{C^0(\partial\Omega \times I)} + |d|_{C^0(\bar{\Omega})})$$

for all $\tau \in [0, 1]$, since $u_\sigma = 0$ for $\sigma = 0$.

By writing the first equation in (13) in the form

$$Lv + v = a(|\nabla u_\sigma|^2 - |\nabla v - \nabla u_\sigma|^2) + a(\sigma - \tau), \quad (15)$$

it follows from L^p -estimates for linear parabolic partial differential equations (see e.g. Ladyženskaja *et al.* [20, Chap. IV, §9]) that there is a constant $\gamma_1 > 0$ independent of v such that

$$|v|_{W_p^{2,1}(\Omega \times I)} \leq \gamma_1 (|Lv + v|_{L^p(\Omega \times I)} + |d|_{W_p^{2-2/p}(\Omega)} + |b|_{W^{1-1/p, (1-1/p)/2}(\partial\Omega \times I)}). \quad (16)$$

Moreover, by using Cauchy–Schwarz inequality (in \mathbb{R}^N), the inequality $2\alpha\beta \leq \alpha^2 + \beta^2$ and the triangle inequality in the right hand side of Eq. (15), one has

$$\begin{aligned} |Lv + v|_{L^p(\Omega \times I)} &\leq 2 |a|_{C^0(\bar{\Omega} \times I)} |\nabla v|^2|_{L^p(\Omega \times I)} \\ &\quad + 3 |a|_{C^0(\bar{\Omega} \times I)} |\nabla u_\sigma|^2|_{L^p(\Omega \times I)} + |a|_{C^0(\bar{\Omega} \times I)}. \end{aligned} \quad (17)$$

Now, by the triangle inequality and the definition of $|\cdot|_{L^p(\Omega \times I)}$ (see e.g. Ladyženskaja *et al.* [20, p. 4]), one has

$$\begin{aligned} |\nabla v|^2|_{L^p(\Omega \times I)} &= \left| \sum_{i=1}^N \left| \frac{\partial v}{\partial x_i} \right|^2 \right|_{L^p(\Omega \times I)} \\ &\leq \sum_{i=1}^N \left| \left| \frac{\partial v}{\partial x_i} \right|^2 \right|_{L^p(\Omega \times I)} = \sum_{i=1}^N \left(\int_0^T \left(\left| \frac{\partial v}{\partial x_i}(\cdot, t) \right|_{L^{2p}(\Omega)} \right)^{2p} dt \right)^{1/p} \\ &\leq \sum_{i=1}^N \left(\int_0^T \left[\left| \frac{\partial v}{\partial x_i}(\cdot, t) \right|_{L^{2p}(\Omega)}^2 \right]^p dt \right)^{1/p}. \end{aligned}$$

Therefore, by the interpolation inequality in Theorem 10.1 in Friedman [13, pp. 27–28, Part I, Sect. 10], one gets

$$\begin{aligned} |\nabla v|^2|_{L^p(\Omega \times I)} &\leq \sum_{i=1}^N \left(\int_0^T [C |v(\cdot, t)|_{W_p^2(\Omega)} |v(\cdot, t)|_{L^\infty(\Omega)}]^p dt \right)^{1/p} \\ &\leq CN \left(\int_0^T [|v(\cdot, t)|_{W_p^2(\Omega)} |v(\cdot, t)|_{L^\infty(\Omega)}]^p dt \right)^{1/p} \\ &\leq CN |v|_{C^0(\bar{\Omega} \times I)} \left(\int_0^T |v(\cdot, t)|_{W_p^2(\Omega)}^p dt \right)^{1/p}, \end{aligned}$$

where $C > 0$ is a constant independent of v . (With the notations in Friedman [13], the constant C depends only on Ω , $m = 2$, $j = 1$, $a = 1/2$, $q = \infty$ and $r = p$.)

Consequently,

$$|\nabla v|^2|_{L^p(\Omega \times I)} \leq \gamma_2 |v|_{C^0(\Omega \times I)} |v|_{W_p^{2,1}(\Omega \times I)}$$

for some constant $\gamma_2 > 0$ independent of v ; which implies that

$$||\nabla u_\sigma|^2|_{L^p(\Omega \times I)} \leq \gamma_2 |u_\sigma|_{C^0(\bar{\Omega} \times I)} |u_\sigma|_{W_p^{2,1}(\Omega \times I)}.$$

Thus, by the estimate (14), one has

$$||\nabla v|^2|_{L^p(\Omega \times I)} \leq \gamma_2 |\sigma - \tau| (|a|_{C^0(\bar{\Omega} \times I)} + |b|_{C^0(\partial\Omega \times I)} + |d|_{C^0(\bar{\Omega})}) |v|_{W_p^{2,1}(\Omega \times I)};$$

which also implies that

$$||\nabla u_\sigma|^2|_{L^p(\Omega \times I)} \leq \sigma \gamma_2 (|a|_{C^0(\bar{\Omega} \times I)} + |b|_{C^0(\partial\Omega \times I)} + |d|_{C^0(\bar{\Omega})}) |u_\sigma|_{W_p^{2,1}(\Omega \times I)}.$$

It then follows from inequalities (16) and (17) that

$$\begin{aligned} |v|_{W_p^{2,1}(\Omega \times I)} &\leq 6\sigma\gamma_1\gamma_2 |a|_{C^0(\bar{\Omega} \times I)} (|a|_{C^0(\bar{\Omega} \times I)} + |b|_{C^0(\partial\Omega \times I)} + |d|_{C^0(\bar{\Omega})}) |u_\sigma|_{W_p^{2,1}(\Omega \times I)} \\ &\quad + 2\gamma_1 (|a|_{C^0(\bar{\Omega} \times I)} + |d|_{W_p^{2-2/p}(\Omega)}) + 2\gamma_1 |b|_{W_p^{1-1/p, (1-1/p)/2}(\partial\Omega \times I)} \end{aligned} \quad (18)$$

provided $|\sigma - \tau| \leq \varepsilon$ where

$$\varepsilon = [4(1 + \gamma_1\gamma_2 |a|_{C^0(\bar{\Omega} \times I)} (|a|_{C^0(\bar{\Omega} \times I)} + |b|_{C^0(\partial\Omega \times I)} + |d|_{C^0(\bar{\Omega})}))]^{-1}.$$

Since $v = u_\sigma - u_\tau$ (and $u_\sigma = 0$ for $\sigma = 0$) one immediately deduces from the inequality (18) that

$$|u_\tau|_{W_p^{2,1}(\Omega \times I)} \leq 2\gamma_1 (1 + |a|_{C^0(\bar{\Omega} \times I)} + |d|_{W_p^{2-2/p}(\Omega)}) (1 + |b|_{W_p^{1-1/p, (1-1/p)/2}(\partial\Omega \times I)})$$

for every $\tau \in [0, 1]$ with $\tau \leq \varepsilon$.

Hence, the inequality (18) implies an *a priori* estimate of the form

$$\begin{aligned} |u_\tau|_{W_p^{2,1}(\Omega \times I)} &\leq \gamma_3 (|a|_{C^0(\bar{\Omega} \times I)} + |b|_{C^0(\partial\Omega \times I)} + \\ &\quad |d|_{W_p^{2-2/p}(\Omega \times I)}) (1 + |b|_{W_p^{1-1/p, (1-1/p)/2}(\partial\Omega \times I)}) \end{aligned}$$

for every $\tau \in [\varepsilon, \min\{1, 2\varepsilon\}]$, where $\gamma_3 = \gamma_3(\cdot, \cdot, \cdot)$ is a constant depending only on its arguments. Obviously this estimate also holds for every $\tau \in [0, \min\{1, 2\varepsilon\}]$.

Thus, by repeating this argument for a finite number of steps, the inequality (10) in the conclusion of Lemma 3.1 follows. The proof is complete. \square

Let us observe that, under the conditions in Lemma 3.1, if $u \in W_p^{2,1}(\Omega \times I)$ is a solution to Eq. (9) (with the first relation in (9) satisfied in the a.e. sense), then u also is a classical solution. Indeed, this follows from the continuous imbedding of $W_p^{2,1}(\Omega \times I)$ into $C^{1+\mu, (1+\mu)/2}(\bar{\Omega} \times I)$ and (interior) regularity of solutions to a parabolic partial differential equation (see

Theorem 9 in Chap. 3, Sect. 4 in Friedman [12, pp. 69–71] or Theorem 12.2 in Chap. 3, §12 in Ladyženskaja *et al.* [20, p. 224]), since the initial condition d satisfies the compatibility condition of order zero.

The following interpolation inequalities will be needed in the sequel. (It should be noted that these inequalities actually hold for every $p \geq 1$ with $p < \infty$.) They will especially be needed for parabolic problems with non-linear boundary conditions.

LEMMA 3.2. *There is a constant $C > 0$ such that for all $u \in W_p^{2,1}(\Omega \times I)$ one has*

$$|u|_{W_p^{1,1/2}(\Omega \times I)} \leq C |u|_{W_p^{2,1}(\Omega \times I)}^{1/2} |u|_{L^p(\Omega \times I)}^{1/2}. \quad (19)$$

Moreover, for every $\varepsilon > 0$ there exists a constant $C_\varepsilon > 0$ such that

$$|u|_{W_p^{1,1/2}(\Omega \times I)} \leq \frac{\varepsilon}{2} C |u|_{W_p^{2,1}(\Omega \times I)} + C_\varepsilon |u|_{L^p(\Omega \times I)}. \quad (20)$$

Proof. By definition of $|\cdot|_{W_p^{1,1/2}(\Omega \times I)}$ (see e.g. [20, p. 81]) we have that

$$|u|_{W_p^{1,1/2}(\Omega \times I)} = |u|_{W_p^{1,0}(\Omega \times I)} + \langle\langle u \rangle\rangle_{p,t,\Omega \times I}^{(1/2)}$$

where

$$|u|_{W_p^{1,0}(\Omega \times I)} = |u|_{L^p(\Omega \times I)} + \sum_{i=1}^N \left| \frac{\partial u}{\partial x_i} \right|_{L^p(\Omega \times I)}$$

and

$$\langle\langle u \rangle\rangle_{p,t,\Omega \times I}^{(1/2)} = \left(\int_{\Omega} \left[\int_I \int_I \frac{|u(x,t) - u(x,s)|^p}{|t-s|^{1+p/2}} ds dt \right] dx \right)^{1/p}.$$

Note that

$$\langle\langle u \rangle\rangle_{p,t,\Omega \times I}^{(1/2)} \leq \left(\int_{\Omega} |u(x, \cdot)|_{W_p^{1/2}(I)}^p dx \right)^{1/p}$$

where

$$|z|_{W_p^{1/2}(I)} = |z|_{L^p(I)} + \left(\int_I \int_I \frac{|z(t) - z(s)|^p}{|t-s|^{1+p/2}} ds dt \right)^{1/p}.$$

By the interpolation inequality in Theorem 10.1 in Friedman [13, pp. 27–28, Part I, Sect. 10] (also see Theorem 4.17 in Adams [1, p. 79]) and Cauchy–Schwarz inequality, one gets, as in the proof of Lemma 3.1, that

$$\sum_{i=1}^N \left| \frac{\partial u}{\partial x_i} \right|_{L^p(\Omega \times I)} \leq C_1 |u|_{W_p^{2,0}(\Omega \times I)}^{1/2} |u|_{L^p(\Omega \times I)}^{1/2}$$

for some constant $C_1 > 0$ independent of u . Therefore,

$$|u|_{W_p^{1,0}(\Omega \times I)} \leq (1 + C_1) |u|_{W_p^{2,0}(\Omega \times I)}^{1/2} |u|_{L^p(\Omega \times I)}^{1/2}. \quad (21)$$

Furthermore, by the (equivalent) definitions and properties of the fractional order space $W_p^{1/2}(I)$ given in Adams [1, pp. 204–214], the interpolation result given in Lemma 7.16 (b) of [1, pp. 187–188] (also see Peetre [26]), Cauchy–Schwarz inequality and Fubini's theorem, it follows that

$$\begin{aligned} \left(\int_{\Omega} |u(x, \cdot)|_{W_p^{1/2}(I)}^p dx \right)^{1/p} &\leq C_2 \left[\int_{\Omega} |u(x, \cdot)|_{W_p^{2,0}(I)}^{p/2} |u(x, \cdot)|_{L^p(I)}^{p/2} dx \right]^{1/p} \\ &\leq C_2 \left(\int_{\Omega} |u(x, \cdot)|_{W_p^{1,0}(I)}^p dx \right)^{1/2p} \\ &\quad \times \left(\int_{\Omega} |u(x, \cdot)|_{L^p(I)}^p dx \right)^{1/2p} \\ &\leq C_2 |u|_{W_p^{0,1}(\Omega \times I)}^{1/2} |u|_{L^p(\Omega \times I)}^{1/2} \end{aligned}$$

for some constant $C_2 > 0$ independent of u ; which implies that

$$\langle\langle u \rangle\rangle_{p,t,\Omega \times I}^{(1/2)} \leq C_2 |u|_{W_p^{0,1}(\Omega \times I)}^{1/2} |u|_{L^p(\Omega \times I)}^{1/2}. \quad (22)$$

Combining inequalities (21) and (22) one deduces the interpolation inequality (19) with $C = (1 + C_1 + C_2)$. Finally, inequality (20) follows immediately from (19) by Young's inequality or Cauchy–Schwarz inequality with ε (see e.g. [20, p. 58]). The proof is complete. ▀

One can also define the function space $W_p^{1,1/2}(\Omega \times I)$ as an interpolation space between $W_p^{2,1}(\Omega \times I)$ and $L^p(\Omega \times I)$. In that case, one can also prove the interpolation inequalities (19) and (20) by using abstract interpolation inequalities of Lions–Magenes type.

We are now ready to prove an *a priori* estimate on the $W_p^{2,1}(\Omega \times I)$ -norm of a solution for the nonlinear initial boundary value problem in terms of the supremum norm of such a solution and the norm of the initial function.

PROPOSITION 3.1. *Suppose $u \in W_p^{2,1}(\Omega \times I) \cap C^{2,1}(\Omega \times I)$ is a classical solution to the initial boundary value problem*

$$\begin{aligned} Lu + u &= f(x, t, u, \nabla u) && \text{in } \Omega \times (0, T], \\ u + \frac{\partial u}{\partial \eta} &= h(x, t, u) && \text{for all } (x, t) \in \partial\Omega \times (0, T], \\ u(x, 0) &= d(x) && \text{for all } x \in \bar{\Omega}, \end{aligned} \quad (23)$$

where $d \in W_p^{2-2/p}(\Omega)$ satisfies the compatibility condition $d(x) + \partial d / \partial \eta(x) = h(x, 0, d(x))$ for all $x \in \partial\Omega$, and f and h satisfy the conditions given in the previous section (with the possible exception of periodicity).

Then there is an increasing function $\gamma_1: \mathbb{R}_+^2 \rightarrow \mathbb{R}_+$ depending only on $f, h, L, \eta, \bar{\Omega} \times I, N$ and p such that

$$|u|_{W_p^{2,1}(\Omega \times I)} \leq \gamma_1(|u|_{C^0(\bar{\Omega} \times I)}, |d|_{W_p^{2-2/p}(\Omega)}). \quad (24)$$

Proof. Since $u \in W_p^{2,1}(\Omega \times I) \subset C^{1+\mu, (1+\mu)/2}(\bar{\Omega} \times I)$ and the function h is (locally) Lipschitz in the sense defined in (6), it follows that the function defined by $b(\cdot, \cdot) = h(\cdot, \cdot, u(\cdot, \cdot))$ belongs to $W_p^{1-1/p, (1-1/p)/2}(\partial\Omega \times I)$.

Moreover,

$$|b|_{W_p^{1-1/p, (-1/p)/2}(\partial\Omega \times I)} \leq \gamma_2(|u|_{C^0(\bar{\Omega} \times I)})(1 + |u|_{W_p^{1,1/2}(\Omega \times I)}), \quad (25)$$

where $\gamma_2: \mathbb{R}_+ \rightarrow \mathbb{R}_+$ is an appropriate increasing function independent of u . Indeed, let $R_1 = |u|_{C^0(\bar{\Omega} \times I)}$, since the function h is Lipschitz on the (compact) set $\partial\Omega \times I \times [-R_1, R_1]$, one has that there is a constant $M = M(\partial\Omega \times I \times [-R_1, R_1]) > 0$ such that

$$|h(x, t, v) - h(y, s, w)| \leq M(|x - y|^2 + |t - s| + |v - w|^2)^{1/2}$$

for all $(x, t, v), (y, s, w) \in \partial\Omega \times I \times [-R_1, R_1]$. This obviously implies that

$$|h(x, t, v)| \leq M|v| + |h(x, t, 0)|$$

for all $(x, t, v) \in \partial\Omega \times I \times [-R_1, R_1]$.

Therefore, by computing the norm in $W_p^{1-1/p, (1-1/p)/2}(\partial\Omega \times I)$ (using local coordinates for $\partial\Omega$), and the triangle inequality, one has

$$\begin{aligned} & |h(\cdot, \cdot, u)|_{W_p^{1-1/p, (1-1/p)/2}(\partial\Omega \times I)} \\ & \leq M|u|_{W_p^{1-1/p, (1-1)/2}(\partial\Omega \times I)} + M(1 + C_1) + |h(\cdot, \cdot, 0)|_{C^0(\partial\Omega \times I)}, \end{aligned}$$

where $C_1 > 0$ is a constant depending only on $\partial\Omega \times I$.

Thus inequality (25) follows, with for instance

$$\gamma_2(|u|_{C^0(\bar{\Omega} \times I)}) = M(C_1 + 1 + C_2) + |h(\cdot, \cdot, 0)|_{C^0(\partial\Omega \times I)}$$

for some constant $C_2 > 0$ independent of u , since the trace operator is continuous from $W_p^{1,1/2}(\Omega \times I)$ onto $W_p^{1-1/p, (1-1/p)/2}(\partial\Omega \times I)$ (see e.g. Lemma 3.4 in [20, p. 82] with $m = 1/2$ and $r = s = 0$). Note that the quantity $\gamma_2(|u|_{C^0(\bar{\Omega} \times I)})$ depends on $|u|_{C^0(\bar{\Omega} \times I)}$ only through the Lipschitz constant $M = M(\partial\Omega \times I \times [-R_1, R_1])$ of the function h .

Now the initial boundary value problem (23) is equivalent to Eq. (9) where $b \in W_p^{1-1/p, (1-1/p)/2}(\partial\Omega \times I)$ is as above and $a \in C^{v, v/2}(\bar{\Omega} \times I)$, with $0 < v \leq \mu^2$, is given by

$$a(\cdot, \cdot) = f(\cdot, \cdot, u, \nabla u)(1 + |\nabla u|^2)^{-1}.$$

By the (at most) quadratic growth condition (5) one has that

$$|a|_{C^0(\bar{\Omega} \times I)} \leq c(|u|_{C^0(\bar{\Omega} \times I)}),$$

where c is the function given in (5).

Thus the conclusion of Proposition 3.1 follows from Lemma 3.1, inequality (25), and the interpolation inequality (20) in Lemma 3.2 where $\varepsilon > 0$ is chosen sufficiently small. The proof is complete. ■

In the following result we prove an “interior” (in time) *a priori* estimate on the $W_p^{2,1}$ -norm of a solution for the nonlinear initial boundary value problem (23) in terms of the supremum norm of such a solution and the “interior” initial time.

PROPOSITION 3.2. *Suppose all conditions in Proposition 3.1 are met. Then there is an increasing function $\gamma_2: \mathbb{R}_+^2 \rightarrow \mathbb{R}_+$ depending only on $f, h, L, \eta, \bar{\Omega} \times I, N$ and p such that for every $t_0 \in (0, T)$ one has*

$$|u|_{W_p^{2,1}(\Omega \times [t_0, T])} \leq \gamma_2(t_0, |u|_{C^0(\bar{\Omega} \times I)}). \quad (26)$$

Proof. Let $a \in C^{v, v/2}(\bar{\Omega} \times I)$ and $b \in W_p^{1-1/p, (1-1/p)/2}(\partial\Omega \times I)$ be as given in the proof of Proposition 3.1. Then Eq. (23) is equivalent to Eq. (9).

Proceeding as in the proof of Lemma 3.1 (with the same notations) we derive the estimate (14). Now let $\xi: [0, T] \rightarrow [0, 1]$ be a (fixed “cut-off”) C^1 -function such that

$$\xi(t) = 0 \text{ for } 0 \leq t \leq t_0/2 \quad \text{and} \quad \xi(t) = 1 \text{ for } t_0 \leq t \leq T.$$

Then the function defined by $w = \xi v$ satisfies the initial boundary value problem

$$Lw + w = \xi(Lv + v) + \xi'v \quad \text{in } \Omega \times (0, T],$$

$$w + \frac{\partial w}{\partial \eta} = \xi b \quad \text{for all } (x, t) \in \partial\Omega \times (0, T],$$

$$w(x, 0) = 0 \quad \text{for all } x \in \bar{\Omega}.$$

Therefore, by using the triangle inequality, the estimate (17) and the definition of ξ , we obtain the estimate

$$\begin{aligned} |Lw + w|_{L^p(\Omega \times I)} &\leq 2|a|_{C^0(\bar{\Omega} \times I)} |\xi| |\nabla v|^2|_{L^p(\Omega \times I)} + 3|a|_{C^0(\bar{\Omega} \times I)} |\xi| |\nabla u_\sigma|^2|_{L^p(\Omega \times I)} \\ &\quad + |a|_{C^0(\bar{\Omega} \times I)} + |\xi'|_{C^0(I)} |v|_{C^0(\Omega \times I)}. \end{aligned}$$

Likewise, as in the proof of Lemma 3.1, we derive the estimates

$$\begin{aligned} &|\xi| |\nabla v|^2|_{L^p(\Omega \times I)} \\ &\leq CN \left(\int_0^T \left[\left| \sqrt{\xi(t)} v(\cdot, t) \right|_{W_p^2(\Omega)} \left| \sqrt{\xi(t)} v(\cdot, t) \right|_{L^\infty(\Omega)} \right]^p dt \right)^{1/p} \\ &= CN \left(\int_0^T [|\xi(t) v(\cdot, t)|_{W_p^2(\Omega)} |v(\cdot, t)|_{L^\infty(\Omega)}]^p dt \right)^{1/p} \\ &\leq CN |v|_{C^0(\bar{\Omega} \times I)} \left(\int_0^T |\xi(t) v(\cdot, t)|_{W_p^2(\Omega)}^p dt \right)^{1/p} \\ &\leq CN |v|_{C^0(\bar{\Omega} \times I)} |w|_{W_p^{2,1}(\Omega \times I)}, \end{aligned}$$

where $C > 0$ is a constant independent of v (and w). Therefore, it follows from inequality (14) that

$$|\xi| |\nabla v|^2|_{L^p(\Omega \times I)} \leq CN |\sigma - \tau| (|a|_{C^0(\Omega \times I)} + |b|_{C^0(\partial\Omega \times I)} + |d|_{C^0(\bar{\Omega})}) |w|_{W_p^{2,1}(\Omega \times I)};$$

which also implies that

$$|\xi| |\nabla u_\sigma|^2|_{L^p(\Omega \times I)} \leq \sigma CN (|a|_{C^0(\bar{\Omega} \times I)} + |b|_{C^0(\partial\Omega \times I)} + |d|_{C^0(\bar{\Omega})}) |\xi u_\sigma|_{W_p^{2,1}(\Omega \times I)}.$$

Since $w(\cdot, 0) = 0$, it follows from L^p -estimates for linear parabolic partial differential equations that there is a constant $C_1 > 0$ independent of w such that

$$|w|_{W_p^{2,1}(\Omega \times I)} \leq C_1 (|Lw + w|_{L^p(\Omega \times I)} + |\xi b|_{W_p^{1-1/p, (1-1/p)/2}(\partial\Omega \times I)}).$$

Furthermore, by computing the norm in $W_p^{1-1/p, (1-1/p)/2}(\partial\Omega \times I)$ as in the proof of Proposition 3.1 and using the triangle inequality, one deduces that

$$\begin{aligned} |\xi b|_{W_p^{1-1/p, (1-1/p)/2}(\partial\Omega \times I)} &\leq |b|_{W_p^{1-1/p, (1-1/p)/2}(\partial\Omega \times I)} \\ &\quad + |h(\cdot, \cdot, \cdot)|_{C^0(\partial\Omega \times I \times [-R_1, R_1])} |\xi|_{W_p^{1-1/p, (1-1/p)/2}(I)}. \end{aligned}$$

Combing these inequalities, we obtain an estimate of the form (18) for w ; that is,

$$\begin{aligned} |w|_{W_p^{2,1}(\Omega \times I)} &\leq \gamma_3(|a|_{C^0(\bar{\Omega} \times I)}, |b|_{C^0(\partial\Omega \times I)}, |d|_{C^0(\bar{\Omega})}) |u_\sigma|_{W_p^{2,1}(\Omega \times I)} + \gamma_4 |a|_{C^0(\bar{\Omega} \times I)} \\ &\quad + \gamma_4 |\xi'|_{C^0(I)} |v|_{C^0(\bar{\Omega} \times I)} + \gamma_4 |b|_{W_p^{1-1/p, (1-1/p)/2}(\partial\Omega \times I)} \\ &\quad + \gamma_4 |h(\cdot, \cdot, \cdot)|_{C^0(\partial\Omega \times I \times [-R_1, R_1])} |\xi|_{W_p^{1-1/p, (1-1/p)/2}(I)} \end{aligned} \quad (27)$$

provided $|\sigma - \tau| \leq \varepsilon = \varepsilon(|a|_{C^0(\bar{\Omega} \times I)}, |b|_{C^0(\partial\Omega \times I)}, |d|_{C^0(\bar{\Omega})}) > 0$.

Now, since $|v|_{W_p^{2,1}(\Omega \times [t_0, T])} = |w|_{W_p^{2,1}(\Omega \times [t_0, T])} \leq |w|_{W_p^{2,1}(\Omega \times I)}$, one can finish the proof exactly as in Lemma 3.1 where one uses inequality (25) and the arguments in the proof of Proposition 3.1.

Let us finally observe that the value $\gamma_2(t_0, |u|_{C^0(\Omega \times I)})$ depends on t_0 only through the (fixed “cut-off”) function $\xi \in C^1(I)$. The proof is complete. ■

The following *a priori* estimate extends to periodic problems with non-homogeneous and nonautonomous linear boundary conditions a similar result proved by Dancer and Hess [10, Lemma 2.1] for periodic problems with homogeneous and autonomous linear boundary conditions. In the statement of the following result, the functions defined on $\bar{\Omega} \times I$ or $\partial\Omega \times I$ are assumed to be extended T -periodically (in time) to the set $\bar{\Omega} \times J$ or $\partial\Omega \times J$ respectively, where $J = [0, 2T]$.

LEMMA 3.3. *For every $a \in C_T^{v, v/2}(\bar{\Omega} \times I)$ where $0 < v \leq \mu$ is a given number, every $b \in W_p^{1-1/p, (1-1/p)/2}(\partial\Omega \times I)$ with $b(x, 0) = b(x, T)$ for all $x \in \partial\Omega$ (extended T -periodically to $\partial\Omega \times J$), one has that the unique classical solution $u \in C^{2,1}(\Omega \times I) \cap W_p^{2,1}(\Omega \times I)$ to the periodic-parabolic problem*

$$\begin{aligned} Lu + u &= a(x, t)(1 + |\nabla u|^2) \quad \text{for all } (x, t) \in \Omega \times [0, T], \\ u + \frac{\partial u}{\partial \eta} &= b(x, t) \quad \text{for all } (x, t) \in \partial\Omega \times [0, T], \\ u(x, 0) &= u(x, T) \quad \text{for all } x \in \bar{\Omega} \end{aligned} \quad (28)$$

satisfies the *a priori* estimate

$$\begin{aligned} |u|_{W_p^{2,1}(\Omega \times I)} &\leq \gamma_3(|a|_{C^0(\bar{\Omega} \times I)}, |b|_{C^0(\partial\Omega \times I)}, |u|_{C^0(\bar{\Omega} \times I)}) \\ &\quad \times (1 + |b|_{W_p^{1-1/p, (1-1/p)/2}(\partial\Omega \times J)}), \end{aligned} \quad (29)$$

where $\gamma_3: \mathbb{R}_+^3 \rightarrow \mathbb{R}_+$ is an increasing function in each argument depending only on $L, \eta, \Omega \times I, N$ and p .

Proof. The uniqueness follows from the arguments used in the first part of the proof of Lemma 3.1, the periodicity of u and the maximum principle (see e.g. [20, 27]) since u is a classical solution.

Assuming, as aforementioned, that a , u , and b , η have been extended T -periodically to the sets $\bar{\Omega} \times J$ and $\partial\Omega \times J$ respectively, it follows that $u \in W_p^{2,1}(\Omega \times J)$ is a (unique) solution to the initial boundary value problem

$$\begin{aligned} Lz + z &= a(x, t)(1 + |\nabla z|^2) & \text{for all } (x, t) \in \Omega \times (0, 2T], \\ z + \frac{\partial z}{\partial \eta} &= b(x, t) & \text{for all } (x, t) \in \partial\Omega \times (0, 2T], \\ z(x, 0) &= d(x) & \text{for all } x \in \bar{\Omega}, \end{aligned} \quad (30)$$

where $d(x) \stackrel{\text{def}}{=} u(x, T)$ since $u(x, T) = u(x, 0)$ is given.

It then follows, from the arguments used in the proof of Proposition 3.2 with $t_0 = T$, (in particular an estimate of the form (27),) that

$$\begin{aligned} |u|_{W_p^{2,1}(\Omega \times [T, 2T])} &\leq \gamma_4(T, |a|_{C^0(\bar{\Omega} \times I)}, |b|_{C^0(\partial\Omega \times I)}, |u|_{C^0(\bar{\Omega} \times I)}) \\ &\quad \times (1 + |b|_{W_p^{1-1/p, (1-1/p)/2}(\partial\Omega \times I)}), \end{aligned} \quad (31)$$

for some nonnegative increasing function γ_4 , where we have used the T -periodicity of the functions u , a , and b .

Setting

$$\begin{aligned} &\gamma_3(|a|_{C^0(\bar{\Omega} \times I)}, |b|_{C^0(\partial\Omega \times I)}, |u|_{C^0(\bar{\Omega} \times I)}) \\ &\quad \equiv \gamma_4(T, |a|_{C^0(\bar{\Omega} \times I)}, |b|_{C^0(\partial\Omega \times I)}, |u|_{C^0(\Omega \times I)}) \end{aligned}$$

and using the T -periodicity of u on the left hand side of inequality (31), the conclusion of Lemma 3.3 follows. The proof is complete. ■

Finally we obtain an *a priori* estimate on the $W_p^{2,1}(\Omega \times I)$ -norm of a solution for periodic-parabolic problems with (possibly) *nonlinear* boundary conditions only in terms of the supremum norm of such a solution.

PROPOSITION 3.3. *There is an increasing function $\gamma_4: \mathbb{R}_+ \rightarrow \mathbb{R}_+$ depending only on f , h , L , η , $\bar{\Omega} \times I$, N and p such that*

$$|u|_{W_p^{2,1}(\Omega \times I)} \leq \gamma_4(|u|_{C^0(\bar{\Omega} \times I)}) \quad (32)$$

for every classical solution $u \in W_p^{2,1}(\Omega \times I) \cap C^{2,1}(\Omega \times I)$ to the periodic-parabolic problem

$$\begin{aligned} Lu + u &= f(x, t, u, \nabla u) & \text{in } \Omega \times [0, T], \\ u + \frac{\partial u}{\partial \eta} &= h(x, t, u) & \text{for all } (x, t) \in \partial\Omega \times [0, T], \\ u(x, 0) &= u(x, T) & \text{for all } x \in \bar{\Omega}, \end{aligned} \quad (33)$$

where f and h satisfy the conditions given in the previous section.

Proof. Let $a \in C^{v, v/2}(\bar{\Omega} \times I)$, with $0 < v \leq \mu^2$, and $b \in W_p^{1-1/p, (1-1/p)/2}(\partial\Omega \times I)$ be defined as in the proof of Proposition 3.1; that is,

$$a(\cdot, \cdot) = f(\cdot, \cdot, u, \nabla u)(1 + |\nabla u|^2)^{-1}$$

and

$$b(\cdot, \cdot) = h(\cdot, \cdot, u(\cdot, \cdot)).$$

Then Eq. (33) is equivalent to the periodic-parabolic problem (28).

Assuming that a and b have been extended by T -periodicity to $\bar{\Omega} \times J$ and $\partial\Omega \times J$, with $J = [0, 2T]$, it follows from the (at most) quadratic growth condition (5) and Lemma 3.3 that

$$|u|_{W_p^{2,1}(\Omega \times I)} \leq \gamma_3(|u|_{C^0(\bar{\Omega} \times I)})(1 + |b|_{W_p^{1-1/p, (1-1/p)/2}(\partial\Omega \times J)}), \quad (34)$$

where γ_3 is an increasing function depending only on $f, h, L, \eta, \Omega \times I, N$ and p .

Furthermore, by computing the norm of the function b in $W_p^{1-1/p, (1-1/p)/2}(\partial\Omega \times J)$ (using local coordinates for $\partial\Omega$), as in the proof of Proposition 3.1, one deduces an inequality of the form (25) where $\partial\Omega \times I$ and $\Omega \times I$ are replaced by $\partial\Omega \times J$ and $\Omega \times J$ respectively.

Thus, the conclusion of Proposition 3.3 follows from the estimates (34), (25), the interpolation inequality (20) in Lemma 3.2 (with $\Omega \times I$ replaced by $\Omega \times J$) where ε is chosen sufficiently small, and the (time) T -periodicity of the function u . The proof is complete. ■

4. EXISTENCE AND REGULARITY RESULTS

This section is devoted to existence and regularity results for periodic-parabolic equations with nonlinear boundary conditions (Theorems 4.1 and 4.2). At the end of the section we indicate how one can obtain minimal and maximal time-periodic solutions to parabolic problems with nonlinear boundary conditions (Remark 4.2). Remark 4.1 is concerned with a discussion of a more general concept of upper and lower solutions as it applies to our problem.

THEOREM 4.1. Suppose $\alpha \in C^{2,1}(\bar{\Omega} \times [0, T_1])$ is a lower solution and $\beta \in C^{2,1}(\bar{\Omega} \times [0, T_1])$ is an upper solution for Eq. (7) such that

$$\alpha(x, t) \leq \beta(x, t) \quad \text{for all } (x, t) \in \bar{\Omega} \times [0, T_1]. \quad (35)$$

Then Eq. (7) has at least one classical solution $u \in C^{2,1}(\Omega \times I) \cap C^{1,0}(\bar{\Omega} \times I)$ such that

$$\alpha(x, t) \leq u(x, t) \leq \beta(x, t) \quad (36)$$

for all $(x, t) \in \bar{\Omega} \times I$. Actually $u \in C^{1+\mu, (1+\mu)/2}(\bar{\Omega} \times I)$.

Proof. Let $\gamma: \bar{\Omega} \times I \times \mathbb{R} \rightarrow \mathbb{R}$ be a function defined by

$$\gamma(x, t, u) = \max[\alpha(x, t), \min(u, \beta(x, t))]. \quad (37)$$

Then γ is continuous and

$$\alpha(x, t) \leq \gamma(x, t, u) \leq \beta(x, t) \quad \text{for all } (x, t, u) \in \bar{\Omega} \times I \times \mathbb{R}. \quad (38)$$

Moreover, by using the triangle inequality, the definition (37) and inequality (35), one gets

$$|\gamma(x, t, u) - \gamma(y, s, v)| \leq |u - v| + \max(|\alpha(x, t) - \alpha(y, s)|, |\beta(x, t) - \beta(y, s)|) \quad (39)$$

for all $(x, t, u), (y, s, v) \in \bar{\Omega} \times I \times \mathbb{R}$.

We consider the modified periodic boundary value problem

$$Lu + u = f(x, t, \gamma(x, t, u), \nabla u) + \gamma(x, t, u) \quad \text{in } \Omega \times [0, T],$$

$$u + \frac{\partial u}{\partial \eta} = \gamma(x, t, u) + h(x, t, \gamma(x, t, u)) \quad \text{for all } (x, t) \in \partial\Omega \times [0, T], \quad (40)$$

$$u(x, 0) = u(x, T) \quad \text{for all } x \in \bar{\Omega},$$

to which we shall apply Leray–Schauder degree type arguments (see e.g. [22, 23]).

Note that if $u \in C^{2,1}(\Omega \times I) \cap C^{1,0}(\bar{\Omega} \times I)$ is a solution to Eq. (40) such that inequalities (36) hold, then u also is a solution to Eq. (7).

We claim that every possible solution $u \in C^{2,1}(\Omega \times I) \cap C^{1,0}(\bar{\Omega} \times I)$ to Eq. (40) satisfies inequalities (36). Suppose $u \in C^{2,1}(\Omega \times I) \cap C^{1,0}(\bar{\Omega} \times I)$ is a solution to Eq. (40). We shall prove that, for all $(x, t) \in \bar{\Omega} \times [0, T]$, the second inequality in (36) is satisfied. (The proof for the first inequality in (36) is similar.)

Suppose that $u(x, t) > \beta(x, t)$ for some $(x, t) \in \bar{\Omega} \times [0, T]$. Then the function $u - \beta$ has a positive maximum attained at a point $(x_0, t_0) \in \bar{\Omega} \times [0, T]$. By the T -periodicity of u and the fact that β is an upper solution, we can assume without loss of generality that $t_0 \in (0, T]$. Indeed, if $u - \beta$ reaches its maximum value at some point $(x_0, 0)$, it immediately follows that $u - \beta$ also reaches its maximum value at the point (x_0, T) (see the last inequality in (8) as it applies to β). Hence,

$$u(x_0, t_0) - \beta(x_0, t_0) > 0 \quad \text{and} \quad u(x_0, t_0) - \beta(x_0, t_0) \geq u(x, t) - \beta(x, t) \quad (41)$$

for all $(x, t) \in \bar{\Omega} \times [0, T]$.

Assume that $x_0 \in \partial\Omega$, it follows from (41) that

$$\frac{\partial u}{\partial \eta}(x_0, t_0) \geq \frac{\partial \beta}{\partial \eta}(x_0, t_0).$$

Therefore, by using the second relation in Eq. (40), the definition (37), and the fact that the function β is an upper solution, one has

$$\begin{aligned} u(x_0, t_0) &= \gamma(x_0, t_0, u(x_0, t_0)) - \frac{\partial u}{\partial \eta}(x_0, t_0) + h(x_0, t_0, \gamma(x_0, t_0, u(x_0, t_0))) \\ &= \beta(x_0, t_0) - \frac{\partial u}{\partial \eta}(x_0, t_0) + h(x_0, t_0, \beta(x_0, t_0)) \\ &\leq \beta(x_0, t_0) - \frac{\partial \beta}{\partial \eta}(x_0, t_0) + h(x_0, t_0, \beta(x_0, t_0)) \\ &\leq \beta(x_0, t_0). \end{aligned}$$

This is a contradiction with the first inequality in (41). Thus $(x_0, t_0) \in \Omega \times (0, T]$.

Consequently,

$$\nabla u(x_0, t_0) = \nabla \beta(x_0, t_0), \quad \frac{\partial(u - \beta)}{\partial t}(x_0, t_0) \geq 0 \quad \text{and} \quad \left(\frac{\partial^2(u - \beta)}{\partial x_i \partial x_j}(x_0, t_0) \right) \leq 0, \quad (42)$$

where $(\partial^2/\partial x_i \partial x_j)$ is the Hessian matrix of $u - \beta$, with respect to the space variable $x \in \mathbb{R}^N$, at the point (x_0, t_0) . Owing to inequalities (2) and (3), it follows that

$$(L(u - \beta))(x_0, t_0) \geq 0. \quad (43)$$

Hence, by the first relation in (42), the first inequality in (41), the first equation in (40), and the definition (37), one has

$$\begin{aligned} (L\beta)(x_0, t_0) &\leq (Lu)(x_0, t_0) = f(x_0, t_0, \gamma(x_0, t_0, u(x_0, t_0)), \nabla u(x_0, t_0)) \\ &\quad + \gamma(x_0, t_0, u(x_0, t_0)) - u(x_0, t_0) \\ &= f(x_0, t_0, \beta(x_0, t_0), \nabla \beta(x_0, t_0)) \\ &\quad + \beta(x_0, t_0) - u(x_0, t_0) \\ &< f(x_0, t_0, \beta(x_0, t_0), \nabla \beta(x_0, t_0)) \leq (L\beta)(x_0, t_0), \end{aligned}$$

since β is an upper solution. This is a contradiction.

Therefore every possible solution $u \in C^{2,1}(\Omega \times I) \cap C^{1,0}(\bar{\Omega} \times I)$ to Eq. (40) satisfies the second inequality in (36). By using similar arguments, one can show that the first inequality in (36) also is satisfied. Thus every possible solution $u \in C^{2,1}(\Omega \times I) \cap C^{1,0}(\bar{\Omega} \times I)$ to Eq. (40) satisfies inequalities (36), and hence u is also a solution to Eq. (7).

Now, in order to prove that Eq. (40) has at least one solution, we shall apply Leray–Schauder degree arguments to Eq. (40). For that purpose, we consider the homotopy

$$\begin{aligned} Lu + u &= \lambda [f(x, t, \gamma(x, t, u), \nabla u) + \gamma(x, t, u)] \quad \text{for } (x, t) \in \Omega \times [0, T], \\ u + \frac{\partial u}{\partial \eta} &= \lambda [\gamma(x, t, u) + h(x, t, \gamma(x, t, u))] \quad \text{for } (x, t) \in \partial\Omega \times [0, T], \\ u(x, 0) &= u(x, T) \quad \text{for } x \in \bar{\Omega}, \end{aligned} \quad (44)$$

where $\lambda \in [0, 1]$. Clearly Eq. (44) reduces to Eq. (40) when $\lambda = 1$.

We first prove that there exists a constant $R > 0$ (R independent of u and λ) such that

$$|u|_{C^0(\bar{\Omega} \times I)} \leq R \quad (45)$$

for all possible solutions $u \in C^{2,1}(\Omega \times I) \cap C^{1,0}(\bar{\Omega} \times I)$ to Eq. (44).

For that purpose, we show how to obtain an estimate for $\max_{\bar{\Omega} \times I} u(x, t)$. (The obtainment of an estimate for $\min_{\bar{\Omega} \times I} u(x, t)$ is similar.) Note that for $\lambda = 0$ the only solution to Eq. (44) is the trivial solution, and for $\lambda = 1$ it follows from the discussion above that

$$|u|_{C^0(\Omega \times I)} \leq \max_{\bar{\Omega} \times I} (|\alpha(x, t)|, |\beta(x, t)|) \leq R_1$$

for some constant $R_1 > 0$.

Therefore, suppose $\lambda \in (0, 1)$ and let $(x_0, t_0) \in \bar{\Omega} \times I$ be such that $u(x_0, t_0) = \max_{\bar{\Omega} \times I} u(x, t)$. By the T -periodicity of u , we can assume without loss of generality that $(x_0, t_0) \in \bar{\Omega} \times (0, T]$. If $x_0 \in \partial\Omega$, it follows that

$$\frac{\partial u}{\partial \eta}(x_0, t_0) \geq 0.$$

By using the second relation in Eq. (44) and the inequality (38) one has

$$\begin{aligned} u(x_0, t_0) &= -\frac{\partial u}{\partial \eta}(x_0, t_0) + \lambda [\gamma(x_0, t_0, u(x_0, t_0)) \\ &\quad + h(x_0, t_0, \gamma(x_0, t_0, u(x_0, t_0)))] \\ &\leq \lambda [\gamma(x_0, t_0, u(x_0, t_0)) + h(x_0, t_0, \gamma(x_0, t_0, u(x_0, t_0)))] \\ &\leq R_1 + \max_{\partial\Omega \times I \times [-R_1, R_1]} |h(x, t, v)| \stackrel{\text{def}}{=} R_2. \end{aligned}$$

Now, if $(x_0, t_0) \in \Omega \times (0, T]$, then

$$\nabla u(x_0, t_0) = 0, \quad \frac{\partial u}{\partial t}(x_0, t_0) \geq 0, \quad \text{and} \quad \left(\frac{\partial^2 u}{\partial x_i \partial x_j}(x_0, t_0) \right) \leq 0.$$

So, owing to inequalities (2) and (3), one has

$$(Lu)(x_0, t_0) \geq 0.$$

Hence, by the first relation in (44) and the inequality (38), one gets

$$\begin{aligned} u(x_0, t_0) &= -(Lu)(x_0, t_0) + \lambda[f(x_0, t_0, \gamma(x_0, t_0, u(x_0, t_0)), 0) \\ &\quad + \gamma(x_0, t_0, u(x_0, t_0))], \\ &\leq \lambda[f(x_0, t_0, \gamma(x_0, t_0, u(x_0, t_0)), 0) + \gamma(x_0, t_0, u(x_0, t_0))] \\ &\leq \max_{\bar{\Omega} \times I \times [-R_1, R_1]} |f(x, t, v, 0)| + R_1 \stackrel{\text{def}}{=} R_3. \end{aligned}$$

Therefore, $u(x, t) \leq R_4 = \max(R_2, R_3)$ for all $(x, t) \in \bar{\Omega} \times I$. Likewise, one can show that $u(x, t) \geq -R_4$ for all $(x, t) \in \bar{\Omega} \times I$. Thus inequality (45) holds.

By the growth conditions on h and inequality (39) the functions h and γ are respectively locally Lipschitz continuous and Lipschitz continuous in their variables, it follows that they are Lipschitz continuous on the compact set $\bar{\Omega} \times I \times [-R, R]$ where R is given by inequality (45).

Now, since $W_p^{2,1}(\Omega \times I) \subset C^{1+\mu, (1+\mu)/2}(\bar{\Omega} \times I) \subset C^{1,1/2}(\bar{\Omega} \times I)$ with $p = (N+2)/(1-\mu)$ as in the previous section (see e.g. Ladyženskaja *et al.* [20, p. 8] for definition), one has that $[\gamma(\cdot, \cdot, u(\cdot, \cdot)) + h(\cdot, \cdot, \gamma(\cdot, \cdot, u(\cdot, \cdot)))] \in C^{1,1/2}(\partial\Omega \times I)$ for each $u \in W_p^{2,1}(\Omega \times I)$ satisfying inequality (45). Moreover, by the μ -Hölder continuity of the (nonlinear) function f and the (interior) regularity properties of solutions to linear parabolic partial differential equations (see e.g. Friedman [12, pp. 69–71], [13] and Ladyženskaja *et al.* [20, p. 224]), one has that every possible solution $u \in W_p^{2,1}(\Omega \times I)$ to Eq. (44) actually belongs to $C^{2,1}(\Omega \times I) \cap C^{1,0}(\bar{\Omega} \times I)$; that is, it is a classical solution.

Hence, in order to apply Leray–Schauder degree techniques to Eq. (44), we will show that all possible solutions $u \in W_p^{2,1}(\Omega \times I)$ to Eq. (44) are (uniformly) bounded independently of λ ; that is, we claim that there exists a constant $\rho > 0$ such that

$$|u|_{W_p^{2,1}(\Omega \times I)} \leq \rho \tag{46}$$

for all possible solutions $u \in W_p^{2,1}(\Omega \times I)$ to Eq. (44). (ρ is independent of u and λ).

Indeed, let $u \in W_p^{2,1}(\Omega \times I)$ be a solution to Eq. (44) for some λ . Then it follows from Proposition 3.3 that there is an increasing function $\gamma_4: \mathbb{R}_+ \rightarrow \mathbb{R}_+$, independent of u and λ , such that

$$|u|_{W_p^{2,1}(\Omega \times I)} \leq \gamma_4(|u|_{C^0(\bar{\Omega} \times I)}).$$

Note that γ_4 depends only on f (through the function c in (5) since u satisfies inequality (45)), the function h , the lower and upper solutions α and β (through the function γ in (37)–(39)), and $L, \eta, \bar{\Omega} \times I, N$ and p . Since u satisfies inequality (45), we derive immediately the inequality (46).

Now that the necessary *a priori* estimates have been obtained, we shall show that Leray–Schauder degree arguments (see e.g. [22, 23]) apply to the homotopy (44).

First of all note that, by the maximum principle [12, 20, 27], the truncation argument used in the proof of Lemma 3.3, estimates for linear parabolic partial differential equations [12, 20], and an elementary application of the Schauder's fixed point theorem in $C_T^{1+\nu, (1+\nu)/2}(\bar{\Omega} \times I)$ with $0 < \nu < \mu$ fixed, it follows that for every $a \in C_T^{\nu, \nu/2}(\bar{\Omega} \times I)$, the linear periodic-parabolic problem

$$Lu + u = a(x, t) \quad \text{for all } (x, t) \in \Omega \times [0, T],$$

$$u + \frac{\partial u}{\partial \eta} = 0 \quad \text{for all } (x, t) \in \partial\Omega \times [0, T],$$

$$u(x, 0) = u(x, T) \quad \text{for all } x \in \Omega,$$

has a unique solution $u \in C_T^{2+\nu, 1+\nu/2}(\bar{\Omega} \times I)$, denoted $S(a)$, such that $|u|_{C^0(\bar{\Omega} \times I)} \leq |a|_{C^0(\bar{\Omega} \times I)}$. (Actually, one can even use a successive approximation method, as in [21] or [4, Section 2, pp. 289–290], to prove the existence part. Also see [3] for the case when the vector field η is independent of t .) Moreover S is a compact linear operator from $C_T^{\nu, \nu/2}(\bar{\Omega} \times I)$ into $C_T^{1+\nu, (1+\nu)/2}(\bar{\Omega} \times I)$ by the compact imbedding of $C^{2+\nu, 1+\nu/2}(\bar{\Omega} \times I)$ into $C^{1+\nu, (1+\nu)/2}(\bar{\Omega} \times I)$.

Furthermore, by Lemma 3.3 herein (see estimate (29)) and an elementary application of the Schauder's fixed point theorem in $C_T^{1+\nu, (1+\nu)/2}(\bar{\Omega} \times I)$, it follows that for every $b \in W_p^{1-1/p, (1-1/p)/2}(\partial\Omega \times I)$, with $b(x, 0) = b(x, T)$ for all $x \in \partial\Omega$, the linear periodic-parabolic problem with nonhomogeneous linear boundary conditions

$$Lu + u = 0 \quad \text{for all } (x, t) \in \Omega \times [0, T],$$

$$u + \frac{\partial u}{\partial \eta} = b(x, t) \quad \text{for all } (x, t) \in \partial\Omega \times [0, T],$$

$$u(x, 0) = u(x, T) \quad \text{for all } x \in \bar{\Omega},$$

has a unique solution $u \in W_p^{2,1}(\Omega \times I) \cap C^{2,1}(\Omega \times I)$, denoted $B(b)$, such that $|u|_{C^0(\bar{\Omega} \times I)} \leq |b|_{C^0(\partial\Omega \times I)}$. (With the help of Lemma 3.3 one can even use a successive approximation method, as in the last part of the proof of Lemma 3.1 herein, to prove the existence part. Also see [4, Section 2, pp. 289–290].) Moreover B is a compact linear operator from ${}_TW_p^{1-1/p, (1-1/p)/2}(\partial\Omega \times I)$ into $C_T^{1+v, (1+v)/2}(\bar{\Omega} \times I)$ by the compact imbedding of $W_p^{2,1}(\Omega \times I)$ into $C^{1+v, (1+v)/2}(\bar{\Omega} \times I)$. (Let us mention that the trace space ${}_TW_p^{1-1/p, (1-1/p)/2}(\partial\Omega \times I)$ denotes the space of functions in $W_p^{1-1/p, (1-1/p)/2}(\partial\Omega \times I) \subset C^0(\partial\Omega \times I)$ that are T -periodic in time.)

Therefore, $u = S(a) + B(b)$ is the unique solution in $W_p^{2,1}(\Omega \times I)$ to the linear nonhomogeneous periodic-parabolic problem

$$\begin{aligned} Lu + u &= a(x, t) & \text{for all } (x, t) \in \Omega \times [0, T], \\ u + \frac{\partial u}{\partial \eta} &= b(x, t) & \text{for all } (x, t) \in \partial\Omega \times [0, T], \\ u(x, 0) &= u(x, T) & \text{for all } x \in \bar{\Omega}. \end{aligned} \quad (47)$$

Moreover, by setting $K(a, b) = S(a) + B(b)$, one has that K is a compact linear operator from $C_T^{v, v/2}(\bar{\Omega} \times I) \times {}_TW_p^{1-1/p, (1-1/p)/2}(\partial\Omega \times I)$ into $C_T^{1+v, (1+v)/2}(\bar{\Omega} \times I)$ by the arguments used in the proof of Lemma 3.3 and the compact imbedding of $W_p^{2,1}(\Omega \times I)$ into $C^{1+v, (1+v)/2}(\bar{\Omega} \times I)$.

Defining the substitution (Nemytski) operators

$$\begin{aligned} F: C_T^{1+v, (1+v)/2}(\bar{\Omega} \times I) &\rightarrow C_T^{v^2, v^2/2}(\bar{\Omega} \times I), \\ H: C_T^{1+v, (1+v)/2}(\bar{\Omega} \times I) &\rightarrow {}_TW_p^{1-1/p, (1-1/p)/2}(\partial\Omega \times I) \end{aligned}$$

by

$$\begin{aligned} (Fv)(\cdot, \cdot) &= f(\cdot, \cdot, \gamma(\cdot, \cdot, v(\cdot, \cdot)), \nabla v(\cdot, \cdot)) + \gamma(\cdot, \cdot, v(\cdot, \cdot)), \\ (Hv)(\cdot, \cdot) &= h(\cdot, \cdot, v(\cdot, \cdot)) + \gamma(\cdot, \cdot, v(\cdot, \cdot)), \end{aligned}$$

it follows that the (nonlinear) operator $S \circ F + B \circ H$ is completely continuous from $C_T^{1+v, (1+v)/2}(\bar{\Omega} \times I)$ into itself since every (possible) solution to Eq. (47) belongs to $W_p^{2,1}(\Omega \times I)$ and the linear operators S and B are compact.

Hence, the homotopy (44) is equivalent to the fixed-point homotopy

$$u = \lambda[(S \circ F) + (B \circ H)] u, \quad (48)$$

in $C_T^{1+v, (1+v)/2}(\bar{\Omega} \times I) \times [0, 1]$ into $C_T^{1+v, (1+v)/2}(\bar{\Omega} \times I)$, to which Leray–Schauder degree arguments apply.

By using inequality (46) and the continuous imbedding of $W_p^{2,1}(\Omega \times I)$ into $C^{1+v, (1+v)/2}(\bar{\Omega} \times I)$, we deduce that there is a constant $\rho_1 > 0$ (independent of u and λ) such that

$$|u|_{C_T^{1+v, (1+v)/2}(\bar{\Omega} \times I)} < \rho_1 \quad (49)$$

for all possible solutions to Eq. (48). (Recall that every possible solution to Eq. (48) actually belongs to $W_p^{2,1}(\Omega \times I)$ by the theory of linear parabolic partial differential equations.)

Thus, by the homotopy invariance of the Leray–Schauder degree (see e.g. [22, 23]), we deduce that

$$d_{LS}(I - [(S \circ F) + (B \circ H)], \mathcal{D}(0, \rho_1), 0) = d_{LS}(I, \mathcal{D}(0, \rho_1), 0) = 1,$$

where d_{LS} denotes the Leray–Schauder degree, and $\mathcal{D}(0, \rho_1)$ denotes the open ball, centered at the origin with radius $\rho_1 > 0$, in $C_T^{1+v, (1+v)/2}(\bar{\Omega} \times I)$.

Finally, The existence property of the degree implies the existence of at least one solution to Eq. (40). The proof is complete. ■

Now, we shall state and prove an existence and *regularity* result for periodic solutions to Eq. (7). Its proof is based on a careful analysis of the regularity of the functions defined on the boundary $\partial\Omega \times I$ and a bootstrap argument.

THEOREM 4.2. *Suppose $h: \bar{\Omega} \times I \times \mathbb{R} \rightarrow \mathbb{R}$ is a locally Lipschitz continuous function which is T -periodic in t such that $h(x, \cdot, u)$ is $[(1 + \mu)/2]$ -Hölder continuous in the variable t , uniformly for (x, u) in bounded sets of $\bar{\Omega} \times \mathbb{R}$; more precisely, for each $u \in \mathbb{R}$ there exist a (closed) interval $U \subset \mathbb{R}$ about u and a number $M > 0$ such that*

$$|h(x, t, v) - h(y, s, w)| \leq M(|x - y|^2 + |t - s|^{1+\mu} + |v - w|^2)^{1/2} \quad (50)$$

for all $(x, t, v), (y, s, w) \in \bar{\Omega} \times I \times U$.

Furthermore, assume $\partial h / \partial u$ and $\partial h / \partial x_i$ exist and are locally μ -Hölder continuous in (x, t, u) as defined in (4).

Then, under the conditions in Theorem 4.1, Eq. (7) has at least one regular solution $u \in C^{2+\mu, 1+\mu/2}(\bar{\Omega} \times I)$ such that inequality (36) holds.

Proof. It follows from Theorem 4.1 that Eq. (7) has at least one classical solution $u \in C^{1+\mu, (1+\mu)/2}(\bar{\Omega} \times I)$ such that inequality (36) holds. Since

$u \in C^{1,1/2}(\bar{\Omega} \times I)$, by using the fact that the function h is locally Lipschitz continuous as defined in (6), one deduces that $h(\cdot, \cdot, u(\cdot, \cdot)) \in C^{\mu, \mu/2}(\bar{\Omega} \times I)$. Moreover, since

$$\frac{\partial}{\partial x_i} [h(\cdot, \cdot, u(\cdot, \cdot))] = \frac{\partial h}{\partial u}(\cdot, \cdot, u(\cdot, \cdot)) \frac{\partial u}{\partial x_i}(\cdot, \cdot) + \frac{\partial h}{\partial x_i}(\cdot, \cdot, u(\cdot, \cdot)),$$

it follows, from the (local) μ -Hölder continuity of the functions $\partial u / \partial x_i$, $\partial h / \partial u$, $\partial h / \partial x_i$, that $\partial / \partial x_i [h(\cdot, \cdot, u(\cdot, \cdot))] \in C^{\mu, \mu/2}(\bar{\Omega} \times I)$. Furthermore, by using inequality (50) and the fact that $u \in C^{1+\mu, (1+\mu)/2}(\bar{\Omega} \times I)$, one has that $h(x, \cdot, u(x, \cdot)) \in C^{(1+\mu)/2}(I)$, uniformly for $x \in \bar{\Omega}$. Therefore, one can easily evaluate

$$|h(\cdot, \cdot, u(\cdot, \cdot))|_{C^{1+\mu, (1+\mu)/2}(\bar{\Omega} \times I)}$$

(see e.g. Ladyženskaja *et al.* [20, pp. 7–8] for an explicit definition) and show that this quantity makes sense and is finite. Thus, $h(\cdot, \cdot, u(\cdot, \cdot)) \in C^{1+\mu, (1+\mu)/2}(\bar{\Omega} \times I)$, which implies that $h(\cdot, \cdot, u(\cdot, \cdot)) \in C^{1+\mu, (1+\mu)/2}(\partial\Omega \times I)$ since $(1+\mu) > 1$ and $\partial\Omega$ is of class $C^{2+\mu}$.

Now, assuming that the functions $u(x, t)$, $h(x, t, u(x, t))$ and $f(x, t, u(x, t), \nabla u(x, t))$ (resp. the vector field $\eta(x, t)$) have been extended by T -periodicity to the set $\bar{\Omega} \times (0, 2T]$ (resp. $\partial\Omega \times (0, 2T]$), let us consider the initial boundary value problem

$$\begin{aligned} Lv &= \xi(\cdot) f(\cdot, \cdot, u, \nabla u) + \xi'(\cdot) u && \text{in } \Omega \times (0, 2T], \\ \frac{\partial v}{\partial \eta} &= \xi(\cdot) h(\cdot, \cdot, u) && \text{on } \partial\Omega \times (0, 2T], \end{aligned} \quad (51)$$

$$v(\cdot, 0) = 0 \quad \text{on } \bar{\Omega},$$

where $\xi \in C^1([0, 2T], [0, 1])$ is a function such that

$$\xi(t) = 0 \quad \text{for } t \in [0, T/4] \quad \text{and} \quad \xi(t) = 1 \quad \text{for } t \in [T/2, 2T].$$

Note that, by the above considerations, the fact that $u \in C^{1+\mu, (1+\mu)/2}(\bar{\Omega} \times [0, 2T])$ and the (local) μ -Hölder continuity of the function f , one has

$$\xi(\cdot) h(\cdot, \cdot, u(\cdot, \cdot)) \in C^{1+\mu, (1+\mu)/2}(\partial\Omega \times [0, 2T]),$$

and

$$[\xi(\cdot) f(\cdot, \cdot, u(\cdot, \cdot), \nabla u(\cdot, \cdot)) + \xi'(\cdot) u(\cdot, \cdot)] \in C^{\mu^2, \mu^2/2}(\bar{\Omega} \times [0, 2T]).$$

Therefore, by the existence and regularity results for solutions to linear parabolic partial differential equations (see e.g. Ladyženskaja *et al.* [20, pp. 320–321, Theorem 5.3, Chap. IV, §5]), it follows that Eq. (51) has a *unique* solution $v \in C^{2+\mu^2, 1+\mu^2/2}(\bar{\Omega} \times [0, 2T])$. Since the function ξu also is a solution to Eq. (51), one has that $v = \xi u$. Thus, $u \in C^{2+\mu^2, 1+\mu^2/2}(\bar{\Omega} \times [T/2, 2T])$, and by periodicity $u \in C^{2+\mu^2, 1+\mu^2/2}(\bar{\Omega} \times [0, 2T])$.

Finally, we use a bootstrap argument. Since $u \in C^{2+\mu^2, 1+\mu^2/2}(\bar{\Omega} \times I)$, one has that $\nabla u \in C^{1, 1/2}(\bar{\Omega} \times I)$. Hence, by using inequality (4), it follows that $f(\cdot, \cdot, u(\cdot, \cdot), \nabla u(\cdot, \cdot)) \in C^{\mu, \mu/2}(\bar{\Omega} \times I)$. Thus, by the aforementioned regularity results for solutions to linear parabolic partial differential equations, one has that $u \in C^{2+\mu, 1+\mu/2}(\bar{\Omega} \times I)$. The proof is complete. ■

Remark 4.1. Theorems 4.1 and 4.2 remain valid under a more general definition of lower and upper solutions; namely, a function $\alpha \in C^0(\bar{\Omega} \times [0, T_1])$ is called a lower solution if it is locally the pointwise maximum of a finite number of functions which satisfy the definition (8), and a function $\beta \in C^0(\bar{\Omega} \times [0, T_1])$ is called an upper solution if it is locally the pointwise minimum of a finite number of functions which satisfy the definition (8) as it applies to an upper solution (see e.g. [6, 24, 30] for an explicit statement of the aforementioned more general definition; of course, the definition in [6] must be extended up to the boundary $\partial\Omega \times [0, T_1]$ in order to include the nonlinear boundary condition herein).

Indeed, it follows from these definitions of α and β , the connectedness and the compactness of $\bar{\Omega} \times [0, T_1]$, that α and β are (locally) Lipschitz continuous on $\bar{\Omega} \times [0, T_1]$. This implies that α and β belong to $C^{1, 1/2}(\bar{\Omega} \times [0, T_1])$. Therefore, by inequality (39), the function $\gamma(\cdot, \cdot, u(\cdot, \cdot)) \in C^{1, 1/2}(\bar{\Omega} \times [0, T])$ for every $u \in C^{1+\mu, (1+\mu)/2}(\bar{\Omega} \times [0, T]) \subset C^{1, 1/2}(\bar{\Omega} \times [0, T])$; which also implies that $\gamma(\cdot, \cdot, u(\cdot, \cdot)) \in W_p^{1-1/p, (1-1/p)/2}(\partial\Omega \times I)$. This is needed in the proof of Theorem 4.1 since one uses Proposition 3.3.

Remark 4.2. In view of Theorem 4.1 and Remark 4.1 one can show that, under the conditions in Theorem 4.1 with Remark 4.1, Eq. (7) actually has a minimal time-periodic solution u_{\min} and a maximal time-periodic solution u_{\max} relative to the pair α and β such that $\alpha \leq u_{\min} \leq u_{\max} \leq \beta$ on $\bar{\Omega} \times I$.

Indeed, it suffices to use Theorem 4.1 with Remark 4.1 herein, the approach patterned after methods employed by Akö as developed in [6, Theorem 4, pp. 215–216], Proposition 3.3 herein and the fact that $W_p^{2, 1}(\Omega \times I)$ is a reflexive Banach space for $p \geq 2$.

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